Fusion procedure for wreath products of finite groups by the symmetric group

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Abstract

Let G be a finite group. A complete system of pairwise orthogonal idempotents is constructed for the wreath product of G by the symmetric group by means of a fusion procedure, that is by consecutive evaluations of a rational function with values in the group ring. This complete system of idempotents is indexed by standard Young multi-tableaux. Associated to the wreath product of G by the symmetric group, a Baxterized form for the Artin generators of the symmetric group is defined and appears in the rational function used in the fusion procedure.

1. Introduction

The fusion procedure for the symmetric group S_n originates in [9] and has then been developed in the more general situation of Hecke algebras [1]. The procedure allows to express a complete set of primitive idempotents, indexed by standard Young tableaux, of the group ring $\mathbb{C}S_n$ via a certain limiting process on a rational function in several variables with values in $\mathbb{C}S_n$.

In [12], an alternative approach of the fusion procedure for the symmetric group has been proposed, and is based on the existence of a maximal commutative set in $\mathbb{C}S_n$ formed by the Jucys–Murphy elements. More precisely, this approach relies on an expression for the primitive idempotents in terms of the Jucys–Murphy elements [10, 13] and transforms this expression into a fusion formula. The evaluations of the variables of the rational function are now consecutive. This version of the fusion procedure has then been generalized to the Hecke algebras of type A [6], to the Brauer algebras [3, 4], to the Birman–Wenzl–Murakami algebras [5], to the complex reflection groups of type $G(m, 1, n)$ [14] and to the Ariki–Koike algebras [15].

Let G be a finite group. The aim of this article is to generalize the fusion procedure, in the spirit of [12], for the wreath product $\widetilde{G}_n$ of the group G by the symmetric group S_n. The representation theory of the group $\widetilde{G}_n$ is well-known, see *e.g.* [8] or [11], and the irreducible representations are labelled by multi-partitions of size n. Analogues of the Jucys–Murphy elements for the group $\widetilde{G}_n$ have been introduced in [16, 17] and used for an inductive approach to the representation theory of the chain of groups $\widetilde{G}_n$ [16].

The irreducible representation $V_{\underline{\lambda}}$ of $\widetilde{G}_n$ corresponding to the multi-partition $\underline{\lambda}$ admits the following decomposition, as a vector space,

$$V_{\underline{\lambda}} = \bigoplus W_{\mathcal{T}} \ ,$$

where the direct sum is over the set of standard Young multi-tableaux of shape $\underline{\lambda}$. The subspaces $W_{\mathcal{T}}$ are common eigenspaces for the Jucys–Murphy elements $\widetilde{j}_1,\dots,\widetilde{j}_n$ of the group $\widetilde{G}_n$. Let $E_{\mathcal{T}}$ be the idempotent of $\mathbb{C}\widetilde{G}_n$ associated to the subspace $W_{\mathcal{T}}$. The idempotents $E_{\mathcal{T}}$, where $\mathcal{T}$ runs through the set of standard multi-tableaux of size n, form together a complete system of pairwise orthogonal idempotents of $\mathbb{C}\widetilde{G}_n$.

We state here the main result of the article.

Theorem. *The idempotent $E_{\mathcal{T}}$ of $\mathbb{C}\widetilde{G}_n$ corresponding to the standard multi-tableau $\mathcal{T}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}} = \frac{1}{\mathsf{F}^G_{\underline{\lambda}}\,\mathsf{F}_{\underline{\lambda}}}\ \Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n})\Big|_{v_i^{(\alpha)}=\xi^{(\alpha)}_{\mathrm{p}_i}\,,\ \substack{i=1,\dots,n\\ \alpha=1,\dots,m}}\ \Big|_{u_1=\mathrm{c}^G_1}\cdots\ \cdots\Big|_{u_n=\mathrm{c}^G_n}\ . \qquad (1)$$

The function Φ appearing in (1) is a rational function in several variables with values in the group ring $\mathbb{C}\widetilde{G}_n$ and $\mathsf{F}^G_{\underline{\lambda}}$, $\mathsf{F}_{\underline{\lambda}}$ are complex numbers.

The eigenvalues of the Jucys–Murphy elements are not sufficient to distinguish between the different subspaces $W_{\mathcal{T}}$. Therefore the commutative family used here for the fusion procedure is formed by the Jucys–Murphy elements $\widetilde{j}_1,\dots,\widetilde{j}_n$, together with an additional set of elements $\{g_i^{(\alpha)}\}$ of $\mathbb{C}\widetilde{G}_n$. The elements $g_i^{(\alpha)}$ are images in $\mathbb{C}\widetilde{G}_n$ of a set of elements $\{g^{(\alpha)}\}$ which linearly span the center of $\mathbb{C}G$ - the indices $i=1,\dots,n$ indicate in which copies of $\mathbb{C}G$ in $\mathbb{C}\widetilde{G}_n$ they belong (see Sections 2 and 3 for precise definitions).

The set of variables is split into two parts: the variables $v_i^{(\alpha)}$ are first evaluated simultaneously at complex numbers $\xi_\nu^{(\alpha)}$ which are eigenvalues of elements $g_i^{(\alpha)}$. These variables correspond to the positions of the nodes of a multi-partition (their places in the m-tuple). Then the variables $u_1,\dots,u_n$ are consecutively evaluated at the eigenvalues of the Jucys–Murphy elements $\widetilde{j}_1,\dots,\widetilde{j}_n$ and are related to the classical contents of the nodes of a multi-partition.

The rational function Φ can be written as the product of two functions, the first one containing the variables $u_1,\dots,u_n$ and the second one containing the variables $v_i^{(\alpha)}$. As in the fusion procedure for the symmetric groups [12], the function related to the contents (that is, containing the variables $u_1,\dots,u_n$) is build up from a "Baxterized" form of the generators of S_n inside $\mathbb{C}\widetilde{G}_n$. However, the Baxterized form used here is a non-trivial generalization, associated to the finite group G, of the usual Baxterization for the symmetric groups.

The coefficient $(\mathsf{F}^G_{\underline{\lambda}}\,\mathsf{F}_{\underline{\lambda}})^{-1}$ appearing in (1) only depends on the multi-partition $\underline{\lambda}$ and not on the standard multi-tableau $\mathcal{T}$ of shape $\underline{\lambda}$. The element $\mathsf{F}_{\underline{\lambda}}$ is the product of the hook lengths of the nodes in the multi-partition $\underline{\lambda}$ and is independent of the group G. The additional factor $\mathsf{F}^G_{\underline{\lambda}}$ depends on the numbers $\xi_\nu^{(\alpha)}$, and in turn on the choice of the set $\{g^{(\alpha)}\}$ of central elements of $\mathbb{C}G$.

In general the subspaces $W_{\mathcal{T}}$ are not one-dimensional, and therefore the idempotent $E_{\mathcal{T}}$ of $\mathbb{C}\widetilde{G}_n$ is not in general primitive. In fact, the subspaces $W_{\mathcal{T}}$ are all one-dimensional if and only if G is an Abelian finite group. In this situation the idempotents $E_{\mathcal{T}}$, where $\mathcal{T}$ runs through the set of standard multi-tableaux of size n, form together a complete system of primitive pairwise orthogonal idempotents of $\mathbb{C}\widetilde{G}_n$.

For an Abelian finite group G, we provide a simplified fusion procedure for the wreath product $\widetilde{G}_n$. This simplified version is obtained by replacing, in the function Φ appearing in (1), the set $\{g^{(\alpha)}\}$ of central elements of $\mathbb{C}G$ by a smaller subset.

If G is the cyclic group of order m then the group $\widetilde{G}_n$ is isomorphic to the complex reflection group of type $G(m,1,n)$. A fusion procedure for the complex reflection group of type $G(m,1,n)$ has been given in [14]. The fusion procedure presented here, in its simplified version, is slightly different than the one in [14], namely the Baxterized forms are different. The Baxterized form used here, when G is the cyclic group of order 2, has been used in [2] for a fusion procedure for the Coxeter groups of type B (that is, the complex reflection groups of type $G(2,1,n)$).

The paper is organized as follows. Sections 2 and 3 contain definitions and notations about the finite group G and the wreath product $\widetilde{G}_n$. The Jucys–Murphy elements for the group $\widetilde{G}_n$ are defined in Section 4. In Section 5 we introduce the Baxterized form, associated to the wreath product $\widetilde{G}_n$, for the Artin generators of the symmetric group and prove that these Baxterized elements satisfy the Yang–Baxter equation with spectral parameters. In Section 6, we recall standard results on the representation theory of the groups $\widetilde{G}_n$ and give the formula for the idempotents $E_{\mathcal{T}}$ in terms of the elements $\widetilde{\jmath}_1,\dots,\widetilde{\jmath}_n$ and $g_i^{(\alpha)}$. The main result of the article, which gives the fusion procedure for the group $\widetilde{G}_n$, is proved in Section 7. In Section 8, we consider the case of an Abelian finite group G and provide a simplified fusion formula in this situation.

Notation.

We denote by $\mathbb{C}H$ the group ring over the complex numbers of a finite group H. For a vector space V, we denote by Id_V the identity operator on V. Symbols $\underline{v}$ and $\underline{v_i}$, for an integer i, stand for m-tuples of variables, namely $\underline{v} := (v^{(1)},\dots,v^{(m)})$ and $\underline{v_i} := (v_i^{(1)},\dots,v_i^{(m)})$;

2. Definitions

Let G be a finite group and let $\{C_1,\dots,C_m\}$ be the set of all its conjugacy classes. We denote by $g^{(1)},\dots,g^{(m)}$ the following central elements of the group ring $\mathbb{C}G$:

$$g^{(\alpha)} := \frac{1}{|C_\alpha|}\sum_{g\in C_\alpha} g\ ,\quad \alpha=1,\dots,m. \tag{2}$$

Let $\rho_1,\dots,\rho_m$ be the pairwise non-isomorphic irreducible representations of G and $W_1,\dots,W_m$ be the corresponding representation spaces of dimensions, respectively, $\mathrm{d}_1,\dots,\mathrm{d}_m$. Denote by $\chi_1,\dots,\chi_m$ the associated irreducible characters. We define complex numbers $\xi_\nu^{(\alpha)}$, $\alpha,\nu=1,\dots,m$, by:

$$\xi_\nu^{(\alpha)} := \frac{1}{\mathrm{d}_\nu}\chi_\nu(g^{(\alpha)})\ ,\quad \alpha,\nu=1,\dots,m. \tag{3}$$

The central elements $g^{(\alpha)}$ act in the irreducible representations of G as multiples of the identity operators, namely:

$$\rho_\nu(g^{(\alpha)}) = \xi_\nu^{(\alpha)} \cdot \mathrm{Id}_{W_\nu} \quad \text{for } \alpha, \nu = 1, \dots, m\,.$$

It is a standard fact that the elements $g^{(\alpha)}$, $\alpha = 1, \dots, m$, span the center of $\mathbb{C}G$ and therefore, if $\nu \neq \nu'$, there exists some $\alpha \in \{1, \dots, m\}$ such that $\xi_\nu^{(\alpha)} \neq \xi_{\nu'}^{(\alpha)}$; that is, the eigenvalues of the elements $g^{(\alpha)}$, $\alpha = 1, \dots, m$, distinguish between the irreducible representations of G.

Functions $g^{(\alpha)}(v)$. For $\alpha = 1, \dots, m$, we define $S^{(\alpha)}$ to be the set formed by the pairwise different numbers among the $\xi_i^{(\alpha)}$, $i = 1, \dots, m$, that is:

$$S^{(\alpha)} := \{\xi_{i_1}^{(\alpha)}, \xi_{i_2}^{(\alpha)}, \dots, \xi_{i_{k_\alpha}}^{(\alpha)}\} \quad \text{with } 1 = i_1 < i_2 < \dots < i_{k_\alpha} \leq m\ ,$$

such that the numbers $\xi_{i_1}^{(\alpha)}, \xi_{i_2}^{(\alpha)}, \dots, \xi_{i_{k_\alpha}}^{(\alpha)}$ are pairwise different and, for any $i \in \{1, \dots, m\}$, we have $\xi_i^{(\alpha)} = \xi_{i_a}^{(\alpha)}$ for some $a \in \{1, \dots, k_\alpha\}$. By construction, we have in $\mathbb{C}G$ that:

$$\prod_{\xi^{(\alpha)} \in S^{(\alpha)}} \left(g^{(\alpha)} - \xi^{(\alpha)}\right) = 0 \quad \text{for } \alpha = 1, \dots, m. \tag{4}$$

We define the following rational functions in v with values in $\mathbb{C}G$:

$$g^{(\alpha)}(v) := \frac{\prod_{\xi^{(\alpha)} \in S^{(\alpha)}} (v - \xi^{(\alpha)})}{v - g^{(\alpha)}} \quad \text{for } \alpha = 1, \dots, m. \tag{5}$$

The rational functions $g^{(\alpha)}(v)$ can be rewritten as polynomial functions in v as follows. Fix $\alpha \in \{1, \dots, m\}$ and let $a_0^{(\alpha)}, a_1^{(\alpha)}, \dots, a_{k_\alpha}^{(\alpha)}$ be the complex numbers defined by

$$\prod_{\xi^{(\alpha)} \in S^{(\alpha)}} (X - \xi^{(\alpha)}) = a_0^{(\alpha)} + a_1^{(\alpha)} X + \dots + a_{k_\alpha}^{(\alpha)} X^{k_\alpha}\ ,$$

where X is an indeterminate. Define the polynomials $\mathfrak{a}_i^{(\alpha)}(v)$, $i = 0, \dots, k_\alpha$, in v by

$$\mathfrak{a}_i^{(\alpha)}(v) = a_i^{(\alpha)} + a_{i+1}^{(\alpha)}\, v + \dots + a_{k_\alpha}^{(\alpha)} v^{k_\alpha - i} \quad \text{for } i = 0, \dots, k_\alpha.$$

Then we have that:

$$g^{(\alpha)}(v) = \sum_{i=0}^{k_\alpha - 1} \mathfrak{a}_{i+1}^{(\alpha)}(v) \big(g^{(\alpha)}\big)^i = \mathfrak{a}_1^{(\alpha)}(v) + \mathfrak{a}_2^{(\alpha)}(v) g^{(\alpha)} + \dots + \mathfrak{a}_{k_\alpha}^{(\alpha)}(v) \big(g^{(\alpha)}\big)^{k_\alpha - 1}\ . \tag{6}$$

Indeed one can directly verify that $(v - g^{(\alpha)}) \displaystyle\sum_{i=0}^{k_\alpha - 1} \mathfrak{a}_{i+1}^{(\alpha)}(v) \big(g^{(\alpha)}\big)^i = \prod_{\xi^{(\alpha)} \in S^{(\alpha)}} (v - \xi^{(\alpha)})$ (in the verification, it is useful to use the recursive relation $\mathfrak{a}_{i+1}^{(\alpha)}(v) = v^{-1}(\mathfrak{a}_i^{(\alpha)}(v) - a_i^{(\alpha)})$ for $i = 0, \dots, k_\alpha - 1$, together with the initial condition $\mathfrak{a}_0^{(\alpha)}(v) = a_0^{(\alpha)} + a_1^{(\alpha)}\, v + \dots + a_{k_\alpha}^{(\alpha)} v^{k_\alpha}$).

Examples.

– If $S^{(\alpha)} = \{\xi^{(\alpha)}\}$ then we have $g^{(\alpha)}(v) = 1$;

– if $S^{(\alpha)} = \{\xi_1^{(\alpha)}, \xi_2^{(\alpha)}\}$ then we have $g^{(\alpha)}(v) = g^{(\alpha)} + v - \xi_1^{(\alpha)} - \xi_2^{(\alpha)}$;

– if $S^{(\alpha)} = \{\xi_1^{(\alpha)}, \xi_2^{(\alpha)}, \xi_3^{(\alpha)}\}$ then we have

$g^{(\alpha)}(v) = \bigl(g^{(\alpha)}\bigr)^2 + (v - \xi_1^{(\alpha)} - \xi_2^{(\alpha)} - \xi_3^{(\alpha)})g^{(\alpha)} + v^2 - v(\xi_1^{(\alpha)} + \xi_2^{(\alpha)} + \xi_3^{(\alpha)}) + \xi_1^{(\alpha)}\xi_2^{(\alpha)} + \xi_1^{(\alpha)}\xi_3^{(\alpha)} + \xi_2^{(\alpha)}\xi_3^{(\alpha)}$.

3. Wreath product

Definitions. Let $G^n := G \times \cdots \times G$, the Cartesian product of n copies of G. The symmetric group S_n on n letters acts on G^n by permuting the n copies of G. We denote the action by $\sigma(a)$, $\sigma \in S_n$ and $a \in G^n$. The wreath product $\widetilde{G}_n$ of the group G by the symmetric group S_n is the semi-direct product $G^n \rtimes S_n$ defined by this action. The group $\widetilde{G}_n$ consists of the elements (a, σ), $a \in G^n$ and $\sigma \in S_n$, with multiplication given by

$$(a,\sigma)\cdot(a',\sigma') = (a\sigma(a'), \sigma\sigma'), \quad a, a' \in G^n \ \text{ and } \ \sigma, \sigma' \in S_n.$$

The groups $\widetilde{G}_n$ form an inductive chain of group:

$$\{1\} \subset \widetilde{G}_1 \subset \widetilde{G}_2 \subset \dots \dots \subset \widetilde{G}_{n-1} \subset \widetilde{G}_n \subset \dots\ , \tag{7}$$

where the subgroup of $\widetilde{G}_n$ isomorphic to $\widetilde{G}_{n-1}$ is formed by the elements of the form (a, σ), $a \in G^{n-1}$ and $\sigma \in S_{n-1}$. This allows to consider elements of $\mathbb{C}\widetilde{G}_{n-1}$ as elements of $\mathbb{C}\widetilde{G}_n$ and we will often do this without mentioning.

We denote by $s_1, \dots, s_{n-1}$ the following elements of $\widetilde{G}_n$:

$$s_i = \Bigl(1_{G^n}\ ,\ \pi_i\Bigr), \quad i = 1, \dots, n-1, \tag{8}$$

where 1_{G^n} is the unit element of G^n and π_i is the transposition of i and $i+1$. The elements $s_1, \dots, s_{n-1}$ satisfy the following relations:

$$\begin{array}{ll} s_i^2 = 1 & \text{for } i = 1, \dots, n-1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i = 1, \dots, n-2, \\ s_i s_j = s_j s_i & \text{for } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1, \end{array} \tag{9}$$

and generate a subgroup of $\widetilde{G}_n$ isomorphic to the symmetric group S_n.

For $j = 1, \dots, n$, let ϖ_j be the injective morphism from $\mathbb{C}G$ to $\mathbb{C}G^n$ defined by

$$\varpi_j(g) := (1_G, \dots, 1_G,\ g\ , 1_G, \dots, 1_G) \quad \text{ for } g \in \mathbb{C}G, \tag{10}$$

where 1_G is the unit element of the group G and, in the right hand side of (10), g is in the j-th position. Let also ι be the natural injective morphism from $\mathbb{C}G^n$ to $\mathbb{C}\widetilde{G}_n$, given by

$$\iota(a) := (a, 1_{S_n}) \quad \text{ for } a \in \mathbb{C}G^n,$$

where 1_{S_n} is the unit element of the symmetric group S_n.

For any $j = 1, \ldots, n$, the composition $\iota \circ \varpi_j$ is an injective morphism from $\mathbb{C}G$ to $\mathbb{C}\widetilde{G}_n$. We will use the following notation:

$$g_j := \iota \circ \varpi_j(g) \quad \text{for } g \in \mathbb{C}G \text{ and } j = 1, \ldots, n. \tag{11}$$

For $j = 1, \ldots, n$, we define, similarly to (5),

$$g_j^{(\alpha)}(v) := \frac{\prod_{\xi^{(\alpha)} \in S^{(\alpha)}} (v - \xi^{(\alpha)})}{v - g_j^{(\alpha)}} \qquad \text{for } \alpha = 1, \ldots, m\,. \tag{12}$$

Now consider the following element of the group ring of $G \times G$:

$$\mathsf{e} := \frac{1}{|G|} \sum_{g \in G} (g, g^{-1})\ . \tag{13}$$

As noticed in [16], the element e satisfies

$$(g, h)\, \mathsf{e} = \mathsf{e}\, (h, g) \qquad \text{for any } g, h \in G\,, \tag{14}$$

and this implies that e acts as follows in the irreducible representations of $G \times G$:

$$\rho_\nu \otimes \rho_{\nu'}(\mathsf{e}) = \frac{\delta_{\nu,\nu'}}{\mathrm{d}_\nu}\, \mathrm{P}_{W_\nu \otimes W_\nu} \qquad \text{for } \nu, \nu' = 1, \ldots, m\,, \tag{15}$$

where $\mathrm{P}_{W_\nu \otimes W_\nu}$ is the permutation operator of the space $W_\nu \otimes W_\nu$ ($\mathrm{P}(u \otimes v) = v \otimes u$ for $u, v \in W_\nu$).

Elements $e_{i,j}$. For any $i, j = 1, \ldots, n$ such that $i \neq j$, the map $\varpi_i \otimes \varpi_j$ is an injective morphism from the group ring of $G \times G$ to $\mathbb{C}G^n$, and thus the composition $\iota \circ (\varpi_i \otimes \varpi_j)$ is an injective morphism from the group ring of $G \times G$ to $\mathbb{C}\widetilde{G}_n$. We define, for $i = 1, \ldots, n$, $e_{i,i} := 1$ and

$$e_{i,j} := \iota\big(\,\varpi_i \otimes \varpi_j(\mathsf{e})\,\big) \qquad \text{for } i, j = 1, \ldots, n \text{ such that } i \neq j, \tag{16}$$

With the notation (11), elements $e_{i,j}$ can be written as

$$e_{i,j} = \frac{1}{|G|} \sum_{g \in G} g_i\, g_j^{-1}\ , \qquad i, j = 1, \ldots, n\ .$$

By construction, we have $e_{i,j} = e_{j,i}$ for $i, j = 1, \ldots, n$, and

$$s_k\, e_{i,j} = e_{\pi_k(i), \pi_k(j)}\, s_k \qquad \text{for } i, j = 1, \ldots, n \text{ and } k = 1, \ldots, n-1\,, \tag{17}$$

where we recall that π_k is the transposition of k and $k+1$.

For $1 \leq i < j \leq n$, the element $\varpi_i \otimes \varpi_j(\mathsf{e})$ acts in the irreducible representations of G^n as follows (this is implied by (15))

$$\rho_{\nu_1} \otimes \cdots \otimes \rho_{\nu_n}\big(\varpi_i \otimes \varpi_j(\mathsf{e})\big) = \frac{\delta_{\nu_i,\nu_j}}{\mathrm{d}_{\nu_i}}\, \mathrm{P}_{i,j}\ , \qquad \text{for any } \nu_1, \ldots, \nu_n \in \{1, \ldots, m\}, \tag{18}$$

where $\mathrm{P}_{i,j}$ is the operator on $W_{\nu_1}\otimes\cdots\otimes W_{\nu_n}$ which permutes the i-th and j-th spaces and acts as the identity operator anywhere else, that is

$$\mathrm{P}_{i,j}(u_1\otimes\dots u_i\dots u_j\dots\otimes u_n)=u_1\otimes\dots u_j\dots u_i\dots\otimes u_n\ ,\qquad u_a\in W_{\nu_a}\ \text{for}\ a=1,\dots,n\,.$$

The following relations are consequences of (16) and (18):

$$\begin{array}{l}e_{i,j}\,e_{k,l}\ =\ e_{k,l}\,e_{i,j}\qquad \text{for}\ 1\le i<j<k<l\le n\ \text{such that}\ k-j>1\,,\\ e_{i,i+1}e_{k,l}=e_{\pi_i(k),\pi_i(l)}e_{i,i+1}\qquad \text{for}\ i=1,\dots,n-1\ \text{and}\ k,l=1,\dots,n\,,\\ e_{i,i+1}\,e_{i+1,i+2}\,e_{i,i+1}\ =\ e_{i+1,i+2}\,e_{i,i+1}\,e_{i+1,i+2}\qquad \text{for}\ i=1,\dots,n-2\,.\end{array}\tag{19}$$

4. Jucys–Murphy elements

The Jucys–Murphy elements $\widetilde{j}_1,\dots,\widetilde{j}_n$ for the group $\widetilde{G}_n$ are defined [16, 17] by the following initial condition and recursion:

$$\widetilde{j}_1=0\quad\text{and}\quad \widetilde{j}_{i+1}=s_i\widetilde{j}_is_i+e_{i,i+1}s_i\quad\text{for}\ i=1,\dots,n-1\,.\tag{20}$$

The Jucys–Murphy elements $\widetilde{j}_1,\dots,\widetilde{j}_n$ form a commutative set of elements,

$$\widetilde{j}_k\,\widetilde{j}_l=\widetilde{j}_l\,\widetilde{j}_k\qquad\text{for}\ k,l=1,\dots,n\,;\tag{21}$$

moreover they satisfy, for $k=1,\dots,n$,

$$\begin{array}{ll}s_i\,\widetilde{j}_k=\widetilde{j}_k\,s_i & \text{for}\ i=1,\dots,n-1\ \text{such that}\ i\neq k-1,k\,,\\ g_l\,\widetilde{j}_k=\widetilde{j}_k\,g_l & \text{for}\ l=1,\dots,n\ \text{and for any}\ g\in G\,.\end{array}\tag{22}$$

Recall that, for $\alpha=1,\dots,m$, the element $g^{(\alpha)}$ is the central element of $\mathbb{C}G$ defined by (2), and that, for $l=1,\dots,n$, the element $g_l^{(\alpha)}$ is the image in $\mathbb{C}\widetilde{G}_n$ of $g^{(\alpha)}$ by the injective morphism $\iota\circ\varpi_l$. The relation (21) and the second relation in (22) imply that the set of elements $\{g_l^{(\alpha)},\ \alpha=1,\dots,m,\ l=1,\dots,n\}$ together with the Jucys–Murphy elements $\widetilde{j}_1,\dots,\widetilde{j}_n$ form a commutative set of elements of $\mathbb{C}\widetilde{G}_n$.

We state here the following Lemma, which will be used later in the proof of the fusion formula in Section 7.

Lemma 1. *For $l=1,\dots,n-1$, we have*

$$\widetilde{j}_l\,s_ls_{l+1}\dots s_{n-1}=s_ls_{l+1}\dots s_{n-1}\,\widetilde{j}_n-\sum_{k=l}^{n-1}s_ls_{l+1}\dots\widehat{s_k}\dots s_{n-1}\,e_{k,n}\ ,\tag{23}$$

where $s_ls_{l+1}\dots\widehat{s_k}\dots s_{n-1}$ stands for the product $s_ls_{l+1}\dots s_{n-1}$ with s_k removed.

Proof. We prove the formula (23) by induction on $n-l$. The basis of induction (for $l=n-1$) is

$$\widetilde{j}_{n-1}s_{n-1}=s_{n-1}\widetilde{j}_n-e_{n-1,n}\ ,$$

which comes immediately from the recurrence relation in (20).

For $l<n-1$,we use $\widetilde{j}_l\, s_l=s_l\,\widetilde{j}_{l+1}-e_{l,l+1}$ to write

$$\widetilde{j}_l\, s_ls_{l+1}\dots s_{n-1}=s_l\,\widetilde{j}_{l+1}\, s_{l+1}\dots s_{n-1}-e_{l,l+1}\, s_{l+1}\dots s_{n-1}\ .$$

The formula (23) follows, using the induction hypothesis and $e_{l,l+1}\, s_{l+1}\dots s_{n-1}=s_{l+1}\dots s_{n-1}\, e_{l,n}$. □

5. Baxterized elements

Recall the standard Artin presentation of the symmetric group S_n: the group S_n is generated by elements $\overline{s}_1,\dots\overline{s}_{n-1}$ with defining relations as in (9), with s_i replaced by $\overline{s}_i$. The standard Baxterization for the generators $\overline{s}_1,\dots,\overline{s}_{n-1}$ of S_n is:

$$\overline{s}_i(c,c'):=\overline{s}_i+\frac{1}{c-c'}\qquad\text{for } i=1,\dots,n-1\,;$$

the parameters c and c' are called *spectral* parameters.

The elements $s_1,\dots,s_{n-1}$, defined by (8), of $\widetilde{G}_n$ generate a subgroup isomorphic to the symmetric group S_n. We define a generalization of the Baxterized elements $\overline{s}_i(c,c')$, associated to the group $\widetilde{G}_n$, as follows:

$$s_i(c,c'):=s_i+\frac{e_{i,i+1}}{c-c'}\qquad\text{for } i=1,\dots,n-1\,. \tag{24}$$

Proposition 2. *The Baxterized elements $s_i(c,c')$ satisfy the Yang–Baxter equation with spectral parameters:*

$$s_i(c,c')s_{i+1}(c,c'')s_i(c',c'')=s_{i+1}(c',c'')s_i(c,c'')s_{i+1}(c,c')\qquad\textit{for } i=1,\dots,n-2\,; \tag{25}$$

they satisfy also

$$\begin{aligned} &s_i(c,c')s_j(d,d')=s_j(d,d')s_i(c,c')\qquad\textit{for } i,j=1,\dots,n-1 \textit{ such that } |i-j|>1\,,\\ &s_i(c,c')s_i(c',c)=1-\frac{e_{i,i+1}^2}{(c-c')^2}\qquad\textit{for } i=1,\dots,n-1\,. \end{aligned} \tag{26}$$

Proof. The first relation in (26) is immediate using that $s_is_j=s_js_i$ and $e_{i,i+1}\,e_{j,j+1}=e_{j,j+1}\,e_{i,i+1}$ for $i,j=1,\dots,n-1$ such that $|i-j|>1$ (see (9) and (19)).

The second relation in (26) follows from a direct calculation in which one uses that $s_i\,e_{i,i+1}=e_{i,i+1}\,s_i$ for $i=1,\dots,n-1$ (see (17)).

To finish the proof of the Proposition, we develop successively both sides of (25) and, using (17), we move in each term all elements $e_{k,l}$ on the left of the elements s_i and s_{i+1}. We compare the two results:

– The cubic terms in the elements s_i and s_{i+1} coincide since $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

– In both sides, the coefficient in front of $s_i s_{i+1}$ is $\dfrac{e_{i+1,i+2}}{c'-c''}$.

– In both sides, the coefficient in front of $s_{i+1} s_i$ is $\dfrac{e_{i,i+1}}{c-c'}$.

– In the left hand side, the coefficient in front of s_i is

$$\frac{e_{i,i+1}\, e_{i+1,i+2}}{(c-c')(c-c'')} + \frac{e_{i,i+2}\, e_{i,i+1}}{(c-c'')(c'-c'')} \ .$$

Using the second line in (19), we transform this expression in $\dfrac{e_{i,i+1}\, e_{i+1,i+2}}{(c-c')(c'-c'')}$, which coincides with the coefficient in front of s_i in the right hand side (we also use (19) here).

– A similar calculation shows that the coefficient in front of s_{i+1} in the right hand side is equal to $\dfrac{e_{i+1,i+2}\, e_{i,i+1}}{(c-c')(c'-c'')}$, which coincides with the coefficient in front of s_{i+1} in the left hand side.

– The remaining terms in the left hand side are

$$\frac{e_{i,i+2}}{c-c''} + \frac{e_{i,i+1}\, e_{i+1,i+2}\, e_{i,i+1}}{(c-c')(c-c'')(c'-c'')} \ ,$$

which coincide with the remaining terms

$$\frac{e_{i,i+2}}{c-c''} + \frac{e_{i+1,i+2}\, e_{i,i+1}\, e_{i+1,i+2}}{(c'-c'')(c-c'')(c-c')}$$

of the right hand side, using the third line in (19). □

Remark. For $i = 1, \dots, n-1$ and $\alpha = 1, \dots, m$, the elements $s_i(c, c')$ and $g_i^{(\alpha)}(v)$ satisfy a certain limit of the reflection equation with spectral parameters (see for example [7]), namely

$$s_i(c,c')\, g_i^{(\alpha)}(c)\, s_i\, g_i^{(\alpha)}(c') = g_i^{(\alpha)}(c')\, s_i\, g_i^{(\alpha)}(c)\, s_i(c,c') \ . \tag{27}$$

Indeed, due to (12) and (26) and the fact that $e_{i,i+1}$ commutes with s_i and $g_i^{(\alpha)}$, the equality (27) is equivalent to

$$s_i(c',c)\,(c-g_i^{(\alpha)})\, s_i\,(c'-g_i^{(\alpha)}) = (c'-g_i^{(\alpha)})\, s_i\,(c-g_i^{(\alpha)})\, s_i(c',c) \ ,$$

which is proved by a straightforward calculation. We skip the details and just indicate that one uses that $g_i^{(\alpha)} e_{i,i+1} = g_{i+1}^{(\alpha)} e_{i,i+1}$, which follows from (14) together with the fact that $g_i^{(\alpha)}$ commutes with $e_{i,i+1}$.

6. Standard m-tableaux and idempotents of $\widetilde{G}_n$

6.1. m-partitions and m-tableaux

1. m-partitions. Let $\lambda \vdash n$ be a partition of n (we shall also say *of size n*), that is, $\lambda = (\lambda_1, \dots, \lambda_k)$, where λ_j, $j = 1, \dots, k$, are positive integers, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and the size of λ is $|\lambda| := \lambda_1 + \dots + \lambda_k = n$.

We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of rows of nodes containing λ_j nodes in the j-th row, $j = 1, \dots, k$; the rows are numbered from top to bottom. Note that the number of nodes in the diagram of λ is equal to n, the size of λ.

Recall that, for a partition λ, a node θ of λ is called *removable* if the set of nodes obtained from λ by removing θ is still a partition. A node θ' not in λ is called *addable* if the set of nodes obtained from λ by adding θ' is still a partition.

An m-partition $\underline{\lambda}$, or a Young m-diagram $\underline{\lambda}$, of size n is an m-tuple of partitions,

$$\underline{\lambda} := (\lambda^{(1)}, \dots, \lambda^{(m)}) ,$$

such that the total number of nodes in the associated Young diagrams is equal to n, that is $|\lambda^{(1)}| + \cdots + |\lambda^{(m)}| = n$.

A pair (θ, k) consisting of a node θ and an integer $k \in \{1, \dots, m\}$ is called an m-node. The integer k is called the position of the m-node.

Let $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ be an m-partition. An m-node $\underline{\theta} = (\theta, k) \in \underline{\lambda}$ is called removable from $\underline{\lambda}$ if the node θ is removable from $\lambda^{(k)}$. An m-node $\underline{\theta}' = (\theta', k') \notin \underline{\lambda}$ is called addable to $\underline{\lambda}$ if the node θ' is addable to $\lambda^{(k')}$. The set of m-nodes removable from $\underline{\lambda}$ is denoted by $\mathcal{E}_-(\underline{\lambda})$ and the set of m-nodes addable to $\underline{\lambda}$ is denoted by $\mathcal{E}_+(\underline{\lambda})$. For example, the removable/addable 3-nodes (marked with -/+) for the 3-partition $(\square\square, \varnothing, \square)$ are

$$\left(\begin{array}{|c|c|c|}\hline & - & + \\ \hline + \\ \cline{1-1}\end{array} \ , \ \begin{array}{|c|}\hline + \\ \hline\end{array} \ , \ \begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1}\end{array} \right)$$

For an m-node $\underline{\theta}$ lying in the line x and the column y of the k-th diagram, we define $\mathrm{p}(\underline{\theta}) := k$ and $\mathrm{c}(\underline{\theta}) := y - x$. The number $\mathrm{p}(\underline{\theta})$ is the position of $\underline{\theta}$ and the number $\mathrm{c}(\underline{\theta})$ is called the classical content of the m-node $\underline{\theta}$.

For an m-partition $\underline{\lambda}$, we define

$$\mathsf{F}^G_{\underline{\lambda}} := \prod_{\underline{\theta}\in\underline{\lambda}} \left(\prod_{\alpha=1}^{m} \Big(\prod_{\substack{\xi^{(\alpha)} \in S^{(\alpha)} \\ \xi^{(\alpha)} \neq \xi^{(\alpha)}_{\mathrm{p}(\underline{\theta})}}} (\xi^{(\alpha)}_{\mathrm{p}(\underline{\theta})} - \xi^{(\alpha)}) \Big) \right) , \tag{28}$$

where the sets $S^{(\alpha)}$ and the numbers $\xi^{(\alpha)}_{\nu}$, for $\alpha, \nu = 1, \dots, m$, are defined in Section 2.

2. Hook length. Let λ be a partition and θ be a node of λ. The hook of θ in λ is the set of nodes of λ consisting of the node θ together with the nodes which lie either under θ in the same column or to the right of θ in the same row; the hook length $h_\lambda(\theta)$ is the cardinality of the hook of θ in λ.

The definition of the hook length is extended to m-partitions and m-nodes as follows. Let $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ be an m-partition and $\underline{\theta} = (\theta, k)$ an m-node of $\underline{\lambda}$. The hook length $h_{\underline{\lambda}}(\underline{\theta})$ of $\underline{\theta}$ in $\underline{\lambda}$ is the hook length of the node θ in the k-th partition of $\underline{\lambda}$, that is

$$h_{\underline{\lambda}}(\underline{\theta}) := h_{\lambda^{(k)}}(\theta) .$$

For an m-partition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$, we define

$$\mathsf{F}_{\underline{\lambda}} := \prod_{\underline{\theta} \in \underline{\lambda}} h_{\underline{\lambda}}(\underline{\theta}) = \prod_{k=1}^{m} \prod_{\theta \in \lambda^{(m)}} h_{\lambda^{(m)}}(\theta) \ . \tag{29}$$

3. Standard m-tableaux. Let $\underline{\lambda}$ be an m-partition of n. An m-tableau of shape $\underline{\lambda}$ is a bijection between the set $\{1, \dots, n\}$ and the set of m-nodes in $\underline{\lambda}$, that is, an m-tableau of shape $\underline{\lambda}$ is obtained by placing the numbers $1, \dots, n$ in the m-nodes of $\underline{\lambda}$. We call the number n the *size* of the m-tableau. An m-tableau is standard if the numbers increase along any row and any column of every diagram in $\underline{\lambda}$.

For a standard m-tableau $\mathcal{T}$, we denote respectively by $\mathrm{c}(\mathcal{T}|i)$ and $\mathrm{p}(\mathcal{T}|i)$ the classical content and the position of the m-node with number i. For example, for the standard 3-tableau $\mathcal{T} = \left(\boxed{1}\boxed{3}\,,\ \varnothing\,,\ \boxed{2}\right)$, we have

$$\mathrm{c}(\mathcal{T}|1) = 0\,,\quad \mathrm{c}(\mathcal{T}|2) = 0\,,\quad \mathrm{c}(\mathcal{T}|3) = 1 \qquad \text{and} \qquad \mathrm{p}(\mathcal{T}|1) = 1\,,\quad \mathrm{p}(\mathcal{T}|2) = 3\,,\quad \mathrm{p}(\mathcal{T}|3) = 1\,.$$

For a standard m-tableau $\mathcal{T}$, we define the *G-content* $\mathrm{c}^G(\mathcal{T}|i)$ of the m-node with number i by

$$\mathrm{c}^G(\mathcal{T}|i) := \frac{\mathrm{c}(\mathcal{T}|i)}{\mathrm{d}_{\mathrm{p}(\mathcal{T}|i)}}\ , \tag{30}$$

where we recall that $\mathrm{d}_1, \dots, \mathrm{d}_m$ are the dimensions of the pairwise non-isomorphic irreducible representations $W_1, \dots, W_m$ of G. For example, for $m = 3$ and for the same standard 3-tableau as above, we have:

$$\mathrm{c}^G(\mathcal{T}|1) = 0\,,\quad \mathrm{c}^G(\mathcal{T}|2) = 0\,,\quad \mathrm{c}^G(\mathcal{T}|3) = \frac{1}{\mathrm{d}_1}\ .$$

Let $\underline{v}$ be an m-tuple of variables, $\underline{v} := (v^{(1)}, \dots, v^{(m)})$. Let N be a non-negative integer, $\underline{\lambda}$ an m-partition of size N and $\mathcal{T}$ a standard m-tableau of shape $\underline{\lambda}$. For brevity, set $\mathrm{c}_i := \mathrm{c}(\mathcal{T}|i)$ and $\mathrm{p}_i := \mathrm{p}(\mathcal{T}|i)$ for $i = 1, \dots, N$. We define

$$F^G_{\mathcal{T}}(\underline{v}) := \Bigl(\prod_{\alpha=1}^{m} \prod_{\substack{\xi^{(\alpha)} \in S^{(\alpha)} \\ \xi^{(\alpha)} \neq \xi^{(\alpha)}_{\mathrm{p}_N}}} \frac{1}{v^{(\alpha)} - \xi^{(\alpha)}} \Bigr)\ , \tag{31}$$

and

$$F_{\mathcal{T}}(u) := \frac{u - \mathrm{c}_N}{u} \prod_{i=1}^{N-1} \frac{(u - \mathrm{c}_i)^2}{(u - \mathrm{c}_i)^2 - \delta_{\mathrm{p}_i, \mathrm{p}_N}}\ , \tag{32}$$

where $\delta_{\mathrm{p}_i, \mathrm{p}_N}$ is the Kronecker delta.

Let $\underline{\mu}$ be the shape of the standard m-tableau obtained from $\mathcal{T}$ by removing the m-node containing the number N. Then $F^G_{\mathcal{T}}(\underline{v})$ is non-singular at $v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_N}$, $\alpha = 1, \dots, m$, and moreover, from (28),

$$F^G_{\mathcal{T}}(\underline{v})\Big|_{v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_N}\,,\ \alpha = 1, \dots, m} = \left(\mathsf{F}^G_{\underline{\lambda}}\right)^{-1} \mathsf{F}^G_{\underline{\mu}}\ . \tag{33}$$

We will also need the following known result [12, 14] concerning the function $F_{\mathcal{T}}(u)$.

Lemma 3. *The rational function $F_{\mathcal{T}}(u)$ is non-singular at $u = \mathrm{c}_N$ and moreover*

$$F_{\mathcal{T}}(u)\Big|_{u=\mathrm{c}_N} = \mathsf{F}_{\underline{\lambda}}^{-1}\mathsf{F}_{\underline{\mu}} \quad . \tag{34}$$

6.2. Idempotents of $\widetilde{G}_n$ corresponding to standard m-tableaux

The irreducible representations of $\widetilde{G}_n$ are indexed by the set of m-partitions of n (see *e.g.* [8] or [11]). Let $\underline{\lambda}$ be an m-partition of n and denote by $V_{\underline{\lambda}}$ the vector space carrying the irreducible representation of $\widetilde{G}_n$ corresponding to $\underline{\lambda}$. Let $\mathcal{T}$ be a standard m-tableau of shape $\underline{\lambda}$. We set

$$W_{\mathcal{T}} := \bigotimes_{i=1}^{n} W_{\mathrm{p}(\mathcal{T}|i)} \ ,$$

where we recall that $W_1, \dots, W_m$ are the irreducible representation spaces of the group G.

Then the vector space $V_{\underline{\lambda}}$ admits the following decomposition:

$$V_{\underline{\lambda}} = \bigoplus W_{\mathcal{T}} \ ,$$

where the direct sum is over the set of standard m-tableaux of shape $\underline{\lambda}$. We denote by $E_{\mathcal{T}}$ the idempotent of $\mathbb{C}\widetilde{G}_n$ corresponding to the subspace $W_{\mathcal{T}}$.

Recall that the set of elements $\{g_k^{(\alpha)},\ \alpha = 1, \dots, m,\ k = 1, \dots, n\}$ together with the Jucys–Murphy elements $\widetilde{j}_1, \dots, \widetilde{j}_n$ form a commutative family of elements of $\mathbb{C}\widetilde{G}_n$ (see Section 4). For any standard m-tableau $\mathcal{T}$ of size n, the subspace $W_{\mathcal{T}}$ is a common eigenspace for this family of elements. Moreover, we have:

$$g_k^{(\alpha)} E_{\mathcal{T}} = E_{\mathcal{T}}\, g_k^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}(\mathcal{T}|k)} E_{\mathcal{T}} \quad \text{for } \alpha = 1, \dots, m \text{ and } k = 1, \dots, n, \tag{35}$$

where the numbers $\xi_i^{\alpha)}$ are defined by formula (3); from the results in [16], we also have

$$\widetilde{j}_k E_{\mathcal{T}} = E_{\mathcal{T}} \widetilde{j}_k = \mathrm{c}^G(\mathcal{T}|k) E_{\mathcal{T}} \quad \text{for } k = 1, \dots, n. \tag{36}$$

For two standard m-tableaux $\mathcal{T}$ and $\mathcal{T}'$ of size n such that $\mathcal{T} \neq \mathcal{T}'$, we have:

− either there exists $\alpha \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$ such that $\xi^{(\alpha)}_{\mathrm{p}(\mathcal{T}|k)} \neq \xi^{(\alpha)}_{\mathrm{p}(\mathcal{T}'|k)}$.

− or there exists $k \in \{1, \dots, n\}$ such that $\mathrm{c}^G(\mathcal{T}|k) \neq \mathrm{c}^G(\mathcal{T}'|k)$.

Indeed, if there is some $k \in \{1, \dots, n\}$ such that $\mathrm{p}(\mathcal{T}|k) \neq \mathrm{p}(\mathcal{T}'|k)$ then $\xi^{(\alpha)}_{\mathrm{p}(\mathcal{T}|k)} \neq \xi^{(\alpha)}_{\mathrm{p}(\mathcal{T}'|k)}$ for some $\alpha \in \{1, \dots, m\}$ (see Section 2). If $\mathrm{p}(\mathcal{T}|k) = \mathrm{p}(\mathcal{T}'|k)$ for any $k = 1, \dots, m$ then, unless $\mathrm{c}^G(\mathcal{T}|k) \neq \mathrm{c}^G(\mathcal{T}'|k)$ for some $k \in \{1, \dots, n\}$, we have that $\mathcal{T}$ and $\mathcal{T}'$ must have the same shape; moreover the m-tableau $\mathcal{T}'$ must be obtained from $\mathcal{T}$ by permuting the entries inside each diagonal of each diagram. As both m-tableaux are standard, we must have $\mathcal{T} = \mathcal{T}'$ which contradicts the assumption. Thus $\mathrm{c}^G(\mathcal{T}|k) \neq \mathrm{c}^G(\mathcal{T}'|k)$ for some $k \in \{1, \dots, n\}$.

Thus, the idempotent $E_{\mathcal{T}}$ can be expressed in terms of the elements $g_k^{(\alpha)},\ \alpha = 1, \dots, m,\ k = 1, \dots, n$ and the Jucys–Murphy elements $\widetilde{j}_1, \dots, \widetilde{j}_n$.

Let $\underline{\lambda}$ be an m-partition of n and $\mathcal{T}$ be a standard m-tableau of shape $\underline{\lambda}$. Denote by $\underline{\theta}$ the m-node of $\mathcal{T}$ containing the number n, and for brevity, set $\mathrm{c}_n^G := \mathrm{c}^G(\underline{\theta})$ and $\mathrm{p}_n = \mathrm{p}(\underline{\theta})$. As the m-tableau $\mathcal{T}$ is standard, the m-node $\underline{\theta}$ of $\underline{\lambda}$ is removable. Let $\mathcal{U}$ be the standard m-tableau obtained from $\mathcal{T}$ by removing $\underline{\theta}$ and let $\underline{\mu}$ be the shape of $\mathcal{U}$.

We have the following inductive formula for $E_{\mathcal{T}}$:

$$E_{\mathcal{T}} = E_{\mathcal{U}} \prod_{\substack{\underline{\kappa}:\ \underline{\kappa}\in\mathcal{E}_+(\underline{\mu}) \\ \mathrm{c}^G(\underline{\kappa})\neq \mathrm{c}_n^G}} \frac{\widetilde{j}_n - \mathrm{c}^G(\underline{\kappa})}{\mathrm{c}_n^G - \mathrm{c}^G(\underline{\kappa})} \prod_{\alpha=1}^{m} \Biggl(\prod_{\substack{\underline{\kappa}:\ \underline{\kappa}\in\mathcal{E}_+(\underline{\mu}) \\ \xi^{(\alpha)}_{\mathrm{p}(\underline{\kappa})}\neq \xi^{(\alpha)}_{\mathrm{p}_n}}} \frac{g_n^{(\alpha)} - \xi^{(\alpha)}_{\mathrm{p}(\underline{\kappa})}}{\xi^{(\alpha)}_{\mathrm{p}_n} - \xi^{(\alpha)}_{\mathrm{p}(\underline{\kappa})}} \Biggr) , \tag{37}$$

where we recall that $\mathcal{E}_+(\underline{\mu})$ is the set of addable m-nodes of the m-partition $\underline{\mu}$; the initial condition is $E_{\mathcal{U}_0} = 1$, where $\mathcal{U}_0$ is the unique standard m-tableau of size 0. Note that the element $E_{\mathcal{U}}$ in (37) is an idempotent of $\mathbb{C}\widetilde{G}_{n-1}$ and we consider it as an element of $\mathbb{C}\widetilde{G}_n$ due to the chain property of the groups $\widetilde{G}_n$ (see (7)).

Let $\{\mathcal{T}_1, \dots, \mathcal{T}_k\}$ be the set of pairwise different standard m-tableaux that can be obtained from $\mathcal{U}$ by adding an m-node containing the number n. Note that $\mathcal{T} \in \{\mathcal{T}_1, \dots, \mathcal{T}_k\}$. We have:

$$E_{\mathcal{U}} = \sum_{i=1}^{k} E_{\mathcal{T}_i} . \tag{38}$$

Consider the following rational function in u and $\underline{v}$ with values in $\mathbb{C}\widetilde{G}_n$

$$E_{\mathcal{U}} \ \frac{u - \mathrm{c}_n^G}{u - \widetilde{j}_n} \ \prod_{\alpha=1}^{m} \frac{v^{(\alpha)} - \xi^{(\alpha)}_{\mathrm{p}_n}}{v^{(\alpha)} - g_n^{(\alpha)}} , \tag{39}$$

and replace $E_{\mathcal{U}}$ by the right hand side of (38). Then formulas (35)–(36) imply that the rational function (39) is non-singular at $u = \mathrm{c}_n^G$ and $v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_n}$, $\alpha = 1, \dots, m$, and moreover,

$$E_{\mathcal{U}} \ \frac{u - \mathrm{c}_n^G}{u - \widetilde{j}_n} \ \prod_{\alpha=1}^{m} \frac{v^{(\alpha)} - \xi^{(\alpha)}_{\mathrm{p}_n}}{v^{(\alpha)} - g_n^{(\alpha)}} \Bigg|_{\substack{u = \mathrm{c}_n^G \\ v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_n},\ \alpha = 1,\dots,m}} = E_{\mathcal{T}} . \tag{40}$$

Remarks. (i) The elements $E_{\mathcal{T}}$, with $\mathcal{T}$ running through the set of standard m-tableaux of size n, form a complete system of pairwise orthogonal idempotents of $\mathbb{C}\widetilde{G}_n$. For a standard m-tableau $\mathcal{T}$, the idempotent $E_{\mathcal{T}}$ is primitive if and only if $W_{\mathrm{p}(\mathcal{T}|i)}$ is one-dimensional for any $i \in \{1, \dots, n\}$. Thus the elements $E_{\mathcal{T}}$ form a complete system of primitive pairwise orthogonal idempotents of $\mathbb{C}\widetilde{G}_n$ if and only if the group G is Abelian.

(ii) The preceding remark can also be expressed as follows: The commutative subalgebra generated by the set $\{g_k^{(\alpha)},\ \alpha = 1, \dots, m,\ k = 1, \dots, n\}$ together with the Jucys–Murphy elements $\widetilde{j}_1, \dots, \widetilde{j}_n$ is a maximal commutative subalgebra of $\mathbb{C}\widetilde{G}_n$ if and only if the group G is Abelian. □

7. Fusion formula

Let

$$\Gamma(\underline{v_1},\dots,\underline{v_n}) := \prod_{i=1}^{n}\prod_{\alpha=1}^{m} g_i^{(\alpha)}(v_i^{(\alpha)}) \ , \tag{41}$$

where the functions $g_i^{(\alpha)}(v_i^{(\alpha)})$, $\alpha = 1,\dots,m$ and $i = 1,\dots,n$, are defined by (12).

Set $\phi_1(u) := 1$ and, for $k = 2,\dots,n$, let

$$\begin{aligned}\phi_k(u_1,\dots,u_{k-1},u) &:= s_{k-1}(u,u_{k-1})\phi_{k-1}(u_1,\dots,u_{k-2},u)s_{k-1}\\ &= s_{k-1}(u,u_{k-1})s_{k-2}(u,u_{k-2})\dots s_1(u,u_1)\cdot s_1\dots s_{k-2}s_{k-1}\ .\end{aligned} \tag{42}$$

We define the following rational function with values in $\mathbb{C}\widetilde{G}_n$:

$$\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n}) := \phi_n(u_1,\dots,u_n)\,\phi_{n-1}(u_1,\dots,u_{n-1})\dots\dots\phi_1(u_1)\,\Gamma(\underline{v_1},\dots,\underline{v_n})\ . \tag{43}$$

Let $\underline{\lambda}$ be an m-partition of size n and $\mathcal{T}$ a standard m-tableau of shape $\underline{\lambda}$. For $i = 1,\dots,n$, we set $\mathrm{c}_i := \mathrm{c}(\mathcal{T}|i)$, $\mathrm{c}_i^G := \mathrm{c}^G(\mathcal{T}|i)$ and $\mathrm{p}_i := \mathrm{p}(\mathcal{T}|i)$. Let also $\mathcal{U}$ be the standard m-tableau obtained from $\mathcal{T}$ by removing the m-node with number n, and let $\underline{\mu}$ be the shape of $\mathcal{U}$.

Theorem 4. *The idempotent $E_{\mathcal{T}}$ of $\mathbb{C}\widetilde{G}_n$ corresponding to the standard m-tableau $\mathcal{T}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}} = \frac{1}{\mathsf{F}^G_{\underline{\lambda}}\,\mathsf{F}_{\underline{\lambda}}}\,\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n})\Big|_{v_i^{(\alpha)}=\xi^{(\alpha)}_{\mathrm{p}_i},\ \substack{i=1,\dots,n\\ \alpha=1,\dots,m}}\Big|_{u_1=\mathrm{c}_1^G}\dots\ \dots\Big|_{u_n=\mathrm{c}_n^G}\ . \tag{44}$$

Proof. Define, for $k = 1,\dots,n$,

$$\widetilde{\phi}_k(u_1,\dots,u_{k-1},u,\underline{v}) := \phi_k(u_1,\dots,u_{k-1},u)\prod_{\alpha=1}^{m} g_k^{(\alpha)}(v^{(\alpha)}) \tag{45}$$

As $g_k^{(\alpha)}$ commutes with s_i if $i < k-1$, we can rewrite the rational function $\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n})$ as

$$\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n}) = \widetilde{\phi}_n(u_1,\dots,u_n,\underline{v_n})\,\widetilde{\phi}_{n-1}(u_1,\dots,u_{n-1},\underline{v_{n-1}})\dots\dots\widetilde{\phi}_1(u_1,\underline{v_1})\ . \tag{46}$$

We prove the Theorem by induction on n. For $n = 0$, there is nothing to prove.
For $n > 0$, we use (46) and the induction hypothesis to rewrite the right hand side of (44) as

$$\Big(\mathsf{F}^G_{\underline{\lambda}}\,\mathsf{F}_{\underline{\lambda}}\Big)^{-1}\mathsf{F}^G_{\underline{\mu}}\,\mathsf{F}_{\underline{\mu}}\cdot\widetilde{\phi}_n(\mathrm{c}_1^G,\dots,\mathrm{c}_{n-1}^G,u_n,\underline{v_n})\,E_{\mathcal{U}}\Big|_{\substack{v_n^{(\alpha)}=\xi^{(\alpha)}_{\mathrm{p}_n},\\ \alpha=1,\dots,m}}\Big|_{u_n=\mathrm{c}_n^G}\ .$$

Now we use the Proposition 5 below to transform this expression into

$$\Big(\mathsf{F}^G_{\underline{\lambda}}\,\mathsf{F}_{\underline{\lambda}}\Big)^{-1}\mathsf{F}^G_{\underline{\mu}}\,\mathsf{F}_{\underline{\mu}}\cdot\Big(F^G_{\mathcal{T}}(\underline{v})F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n}u)\Big)^{-1}\frac{u-\mathrm{c}_n^G}{u-\widetilde{j}_n}\prod_{\alpha=1}^{m}\frac{v^{(\alpha)}-\xi^{(\alpha)}_{\mathrm{p}_n}}{v^{(\alpha)}-g_n^{(\alpha)}}\,E_{\mathcal{U}}\Big|_{\substack{v^{(\alpha)}=\xi^{(\alpha)}_{\mathrm{p}_n},\\ \alpha=1,\dots,m}}\Big|_{u_n=\mathrm{c}_n^G}\ .$$

We recall that $\mathrm{d}_{\mathrm{p}_n}\mathrm{c}_n^G = \mathrm{c}_n$ and we use formulas (33) and (34) concerning the functions $F^G_{\mathcal{T}}(\underline{v})$ and $F_{\mathcal{T}}(u)$, together with formula (40) to conclude. □

Proposition 5. *Assume that $n \geq 1$. We have*

$$F^G_{\mathcal{T}}(\underline{v}) F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n} u)\, \widetilde{\phi}_n(\mathrm{c}^G_1, \dots, \mathrm{c}^G_{n-1}, u, \underline{v})\, E_{\mathcal{U}} \Big|_{\substack{v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_n}, \\ \alpha = 1, \dots, m}} = \frac{u - \mathrm{c}^G_n}{u - \widetilde{j}_n} \prod_{\alpha=1}^{m} \frac{v^{(\alpha)} - \xi^{(\alpha)}_{\mathrm{p}_n}}{v^{(\alpha)} - g^{(\alpha)}_n}\, E_{\mathcal{U}} \Big|_{\substack{v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_n}, \\ \alpha = 1, \dots, m}} . \quad (47)$$

Proof. Notice that, from (12) and (31),

$$F^G_{\mathcal{T}}(\underline{v}) \prod_{\alpha=1}^{m} g^{(\alpha)}_n(v^{(\alpha)}) = \prod_{\alpha=1}^{m} \frac{v^{(\alpha)} - \xi^{(\alpha)}_{\mathrm{p}_n}}{v^{(\alpha)} - g^{(\alpha)}_n} . \quad (48)$$

Define

$$E_{\mathcal{U},\mathrm{p}_n} := \prod_{\alpha=1}^{m} \frac{v^{(\alpha)} - \xi^{(\alpha)}_{\mathrm{p}_n}}{v^{(\alpha)} - g^{(\alpha)}_n} E_{\mathcal{U}} \Big|_{\substack{v^{(\alpha)} = \xi^{(\alpha)}_{\mathrm{p}_n}, \\ \alpha = 1, \dots, m}} ; \quad (49)$$

the element $E_{\mathcal{U},\mathrm{p}_n}$ is an idempotent which is equal to the sum of the idempotents $E_{\mathcal{V}}$, where $\mathcal{V}$ runs through the set of standard m-tableaux obtained from $\mathcal{U}$ by adding an m-node $\underline{\theta}$ with number n such that $\mathrm{p}(\underline{\theta}) = \mathrm{p}_n$.

Now, using (45), (48) and (49), the Proposition is a direct consequence of the Lemma 6 below. □

Lemma 6. *Assume that $n \geq 1$. We have*

$$F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n} u)\, \phi_n(\mathrm{c}^G_1, \dots, \mathrm{c}^G_{n-1}, u)\, E_{\mathcal{U},\mathrm{p}_n} = \frac{u - \mathrm{c}^G_n}{u - \widetilde{j}_n} E_{\mathcal{U},\mathrm{p}_n} . \quad (50)$$

Proof. The left hand side of (50) is

$$F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n} u)(s_{n-1} + \frac{e_{n-1,n}}{u, \mathrm{c}^G_{n-1}}) \dots (s_1 + \frac{e_{1,2}}{u - \mathrm{c}^G_1}) \cdot s_1 \dots s_{n-1} E_{\mathcal{U},\mathrm{p}_n} .$$

For $k = 1, \dots, n-1$, one can verify, using relations (17) and (19), that

$$e_{k,k+1} \cdot (s_{k-1} + \frac{e_{k-1,k}}{u - \mathrm{c}^G_{k-2}}) \dots (s_1 + \frac{e_{1,2}}{u - \mathrm{c}^G_1}) = (s_{k-1} + \frac{e_{k-1,k}}{u - \mathrm{c}^G_{k-2}}) \dots (s_1 + \frac{e_{1,2}}{u - \mathrm{c}^G_1}) \cdot e_{1,k+1} ,$$

and
$$e_{1,k+1} \cdot s_1 \dots s_{n-1} = s_1 \dots s_{n-1} \cdot e_{k,n} .$$

As moreover $e_{k,n} E_{\mathcal{U},\mathrm{p}_n} = 0$ if $\mathrm{p}_k \neq \mathrm{p}_n$ (see (18)), the left hand side of (50) is equal to

$$F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n} u)(s_{n-1} + \frac{\delta_{\mathrm{p}_{n-1},\mathrm{p}_n} e_{n-1,n}}{u - \mathrm{c}^G_{n-1}}) \dots (s_1 + \frac{\delta_{\mathrm{p}_1,\mathrm{p}_n} e_{1,2}}{u - \mathrm{c}^G_1}) \cdot s_1 \dots s_{n-1} E_{\mathcal{U},\mathrm{p}_n} . \quad (51)$$

We prove the Lemma by induction on n. First assume that $\mathrm{p}_i \neq \mathrm{p}_n$ for $i = 1, \dots, n-1$. In this situation, $E_{\mathcal{U},\mathrm{p}_n} = E_{\mathcal{T}}$ and we have $\mathrm{c}^G_n = 0$, $\widetilde{j}_n E_{\mathcal{T}} = 0$, $F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n} u) = 1$ and, due to the formula (51), $F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n} u)\phi_n(\mathrm{c}^G_1, \dots, \mathrm{c}^G_{n-1}, u)\, E_{\mathcal{U},\mathrm{p}_n} = E_{\mathcal{U},\mathrm{p}_n}$. So in this situation, the formula (50) is trivially satisfied; notice that, in particular, the basis of induction (for $n = 1$) has been proved.

Let $n>1$ and assume that there exists $l\in\{1,\dots,n-1\}$ such that $\mathrm{p}_l=\mathrm{p}_n$. Fix l such that $\mathrm{p}_l=\mathrm{p}_n$ and $\mathrm{p}_i\neq\mathrm{p}_n$ for $i=l+1,\dots,n-1$.

Let $\mathcal{V}$ be the standard m-tableau obtained from $\mathcal{U}$ by removing the m-nodes with numbers $l+1,\dots,n-1$ and $\mathcal{W}$ be the standard m-tableau obtained from $\mathcal{V}$ by removing the m-node with number l. Define, similarly to (49),

$$E_{\mathcal{W},\mathrm{p}_l}:=\prod_{\alpha=1}^{m}\frac{v^{(\alpha)}-\xi^{(\alpha)}_{\mathrm{p}_l}}{v^{(\alpha)}-g^{(\alpha)}_l}E_{\mathcal{W}}\Big|_{\substack{v^{(\alpha)}=\xi^{(\alpha)}_{\mathrm{p}_l},\\ \alpha=1,\dots,m}}\ ;$$

As $E_{\mathcal{W}}E_{\mathcal{U}}=E_{\mathcal{U}}$, $E^2_{\mathcal{U},\mathrm{p}_n}=E_{\mathcal{U},\mathrm{p}_n}$ and $\mathrm{p}_l=\mathrm{p}_n$, we have

$$\begin{aligned}E_{\mathcal{W},\mathrm{p}_l}s_ls_{l+1}\dots s_{n-1}E_{\mathcal{U},\mathrm{p}_n}&=s_ls_{l+1}\dots s_{n-1}\cdot\prod_{\alpha=1}^{m}\frac{v^{(\alpha)}-\xi^{(\alpha)}_{\mathrm{p}_l}}{v^{(\alpha)}-g^{(\alpha)}_n}E_{\mathcal{W}}E_{\mathcal{U},\mathrm{p}_n}\Big|_{\substack{v^{(\alpha)}=\xi^{(\alpha)}_{\mathrm{p}_l},\\ \alpha=1,\dots,m}}\\ &=s_ls_{l+1}\dots s_{n-1}E_{\mathcal{U},\mathrm{p}_n}\ .\end{aligned}\qquad(52)$$

Thus, we rewrite (51) as

$$F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n}u)\,s_{n-1}\dots s_{l+1}\big(s_l+\frac{e_{l,l+1}}{u-\mathrm{c}^G_l}\big)\cdot\phi_l(\mathrm{c}^G_1,\dots,\mathrm{c}^G_{l-1},u)E_{\mathcal{W},\mathrm{p}_l}\cdot s_ls_{l+1}\dots s_{n-1}E_{\mathcal{U},\mathrm{p}_n}\ .$$

We use the induction hypothesis to replace $\phi_l(\mathrm{c}^G_1,\dots,\mathrm{c}^G_{l-1},u)E_{\mathcal{W},\mathrm{p}_l}$ by $\big(F_{\mathcal{V}}(\mathrm{d}_{\mathrm{p}_n}u)\big)^{-1}\dfrac{u-\mathrm{c}^G_l}{u-\tilde{j}_l}\ E_{\mathcal{W},\mathrm{p}_l}$, and we use (52) again to obtain for the left hand side of (50):

$$F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n}u)(F_{\mathcal{V}}(\mathrm{d}_{\mathrm{p}_n}u))^{-1}\,s_{n-1}\dots s_{l+1}\big(s_l+\frac{e_{l,l+1}}{u-\mathrm{c}^G_l}\big)\,\frac{u-\mathrm{c}^G_l}{u-\tilde{j}_l}\,s_ls_{l+1}\dots s_{n-1}E_{\mathcal{U},\mathrm{p}_n}\ .$$

Recall that the inverse of $\big(s_l+\dfrac{e_{l,l+1}}{u-\mathrm{c}^G_l}\big)$ is $\big(s_l+\dfrac{e_{l,l+1}}{\mathrm{c}^G_l-u}\big)\Big(1-\dfrac{e^2_{l,l+1}}{(u-\mathrm{c}^G_l)^2}\Big)^{-1}$ according to the second line in (26). We move $s_{n-1}\dots s_{l+1}\big(s_l+\dfrac{e_{l,l+1}}{u-\mathrm{c}^G_l}\big)\,(u-\tilde{j}_l)^{-1}$ to the right hand side of (50) and, using that $\tilde{j}_n$ commutes with $E_{\mathcal{U},\mathrm{p}_n}$, we move $(u-\tilde{j}_n)^{-1}$ from the right hand side of (50) to the left hand side. We finally obtain that the equality (50) is equivalent to:

$$\begin{aligned}&F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n}u)(F_{\mathcal{V}}(\mathrm{d}_{\mathrm{p}_n}u))^{-1}(u-\mathrm{c}^G_l)s_ls_{l+1}\dots s_{n-1}(u-\tilde{j}_n)\,E_{\mathcal{U},\mathrm{p}_n}\\ &\qquad=(u-\mathrm{c}^G_n)(u-\tilde{j}_l)\big(s_l+\frac{e_{l,l+1}}{\mathrm{c}^G_l-u}\big)\Big(1-\frac{e^2_{l,l+1}}{(u-\mathrm{c}^G_l)^2}\Big)^{-1}s_{l+1}\dots s_{n-1}\,E_{\mathcal{U},\mathrm{p}_n}\ ;\end{aligned}\qquad(53)$$

Now notice that $e_{l,l+1}\cdot s_{l+1}\dots s_{n-1}=s_{l+1}\dots s_{n-1}\cdot e_{l,n}$ and that (see (18)) $e^2_{l,n}\,E_{\mathcal{U},\mathrm{p}_n}=\dfrac{1}{\mathrm{d}^2_{\mathrm{p}_n}}\,E_{\mathcal{U},\mathrm{p}_n}$ since $\mathrm{p}_l=\mathrm{p}_n$. Moreover, formula (32) implies, since $\mathrm{p}_l=\mathrm{p}_n$ and $\mathrm{p}_i\neq\mathrm{p}_n$ for $i=l+1,\dots,n-1$,

$$F_{\mathcal{T}}(\mathrm{d}_{\mathrm{p}_n}u)(F_{\mathcal{V}}(\mathrm{d}_{\mathrm{p}_n}u))^{-1}=\frac{\mathrm{d}_{\mathrm{p}_n}u-\mathrm{c}_n}{\mathrm{d}_{\mathrm{p}_n}u-\mathrm{c}_l}\,\frac{(\mathrm{d}_{\mathrm{p}_n}u-\mathrm{c}_l)^2}{(\mathrm{d}_{\mathrm{p}_n}u-\mathrm{c}_l)^2-1}=\frac{u-\mathrm{c}^G_n}{u-\mathrm{c}^G_l}\,\frac{(u-\mathrm{c}^G_l)^2}{(u-\mathrm{c}^G_l)^2-\frac{1}{\mathrm{d}^2_{\mathrm{p}_n}}}.\qquad(54)$$

Therefore, to verify formula (53), it remains to show that

$$s_l s_{l+1}\dots s_{n-1}(u-\widetilde{j}_n)\,E_{\mathcal{U},\mathrm{p}_n} \;=\; (u-\widetilde{j}_l)\big(s_l+\frac{e_{l,l+1}}{\mathrm{c}_l^G-u}\big)s_{l+1}\dots s_{n-1}\,E_{\mathcal{U},\mathrm{p}_n}\ . \tag{55}$$

Rewrite the right hand side of (55) as

$$(u-\widetilde{j}_l)s_l s_{l+1}\dots s_{n-1}\,E_{\mathcal{U},\mathrm{p}_n}+(u-\widetilde{j}_l)\frac{e_{l,l+1}}{\mathrm{c}_l^G-u}s_{l+1}\dots s_{n-1}\,E_{\mathcal{U},\mathrm{p}_n}\ .$$

Then notice that

$$\widetilde{j}_l\,e_{l,l+1}\,s_{l+1}\dots s_{n-1}\,E_{\mathcal{U},\mathrm{p}_n}=s_{l+1}\dots s_{n-1}\widetilde{j}_l\,e_{l,n}\,E_{\mathcal{U},\mathrm{p}_n}\ .$$

We use that $\widetilde{j}_l\,e_{l,n}\,E_{\mathcal{U},\mathrm{p}_n}=\widetilde{j}_l\,E_{\mathcal{U},\mathrm{p}_n}\,e_{l,n}=\mathrm{c}_l^G\,e_{l,n}\,E_{\mathcal{U},\mathrm{p}_n}$ and the Lemma 1 to replace $\widetilde{j}_l s_l s_{l+1}\dots s_{n-1}$; we obtain for the right hand side of (55)

$$\Big(s_l s_{l+1}\dots s_{n-1}(u-\widetilde{j}_n)+\sum_{k=l}^{n-1}s_l s_{l+1}\dots\widehat{s_k}\dots s_{n-1}\,e_{k,n}-s_{l+1}\dots s_{n-1}e_{l,n}\Big)\,E_{\mathcal{U},\mathrm{p}_n}\ .$$

As $e_{k,n}\,E_{\mathcal{U},\mathrm{p}_n}=0$ if $k>l$, the formula (55) is verified. □

Remark. During the proof of the Theorem 4, we transformed the defining formula (43) for the rational function $\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n})$ into the formula (46). The formula (46) is well adapted to the structure of chain with respect to n of the groups $\widetilde{G}_n$, in the sense that, using formula (46), we write

$$\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n})=\widetilde{\phi}_n(u_1,\dots,u_n,\underline{v_n})\Phi^{(n-1)}(u_1,\dots,u_{n-1},\underline{v_1},\dots,\underline{v_{n-1}})\ ,$$

where $\Phi^{(n-1)}(u_1,\dots,u_{n-1},\underline{v_1},\dots,\underline{v_{n-1}})$ is the rational function corresponding to $\widetilde{G}_{n-1}$ (seen as a rational function with values in $\widetilde{G}_n$) and $\widetilde{\phi}_n(u_1,\dots,u_n,\underline{v_n})$ is defined by (45). The formula (46) is often useful for explicit calculations.

Examples. Here we consider the example when G is the symmetric group on 3 letters. To avoid confusion we will keep the notation G for this group, and $\widetilde{G}_n$ for the wreath product of G by the symmetric group S_n. We use the standard cyclic notation for permutations and we denote the elements of G by 1_G, $(1,2)$, $(1,3)$, $(2,3)$, $(1,2,3)$ and $(1,3,2)$. The central elements, defined in (2), of $\mathbb{C}G$ are

$$g^{(1)}:=\frac{1}{3}\Big((1,2)+(1,3)+(2,3)\Big),\quad g^{(2)}:=\frac{1}{2}\Big((1,2,3)+(1,3,2)\Big)\quad\text{and}\quad g^{(3)}:=1_G\ .$$

Let ρ_1,ρ_2,ρ_3 be the pairwise non-isomorphic irreducible representations of G, namely ρ_1 is the trivial representation, ρ_2 is the sign representation and ρ_3 is the two-dimensional irreducible representation of G. The numbers defined in (3) are equal to:

$$\xi_1^{(1)}=1,\ \xi_2^{(1)}=-1,\ \xi_3^{(1)}=0,\qquad \xi_1^{(2)}=1,\ \xi_2^{(2)}=1,\ \xi_3^{(2)}=-\frac{1}{2}\qquad\text{and}\qquad \xi_1^{(3)}=1,\ \xi_2^{(3)}=1,\ \xi_3^{(3)}=1\ .$$

Thus the functions defined by (5) are, in this example,

$$g^{(1)}(v)=\big(g^{(1)}\big)^2+vg^{(1)}+v^2-1\,,\qquad g^{(2)}(v)=g^{(2)}+v-\frac{1}{2}\qquad\text{and}\qquad g^{(3)}(v)=1\ .$$

Recall the notation $g_i^{(\alpha)}$, $\alpha = 1,2,3$ and $i = 1,\dots,n$, for images of $g^{(\alpha)}$ in $\mathbb{C}\widetilde{G}_n$ (see (11)).

For the 3-partition of size 1 $(\Box,\varnothing,\varnothing)$, we have, from formulas (28) and (29), $\mathsf{F}^G_{(\Box,\varnothing,\varnothing)} = 3$ and $\mathsf{F}_{(\Box,\varnothing,\varnothing)} = 1$.

We obtain for the idempotent $E_{\left(\boxed{1},\varnothing,\varnothing\right)}$ of $\mathbb{C}\widetilde{G}_1 (\cong \mathbb{C}G)$:

$$E_{\left(\boxed{1},\varnothing,\varnothing\right)} = \frac{1}{3}\Big(\big(g_1^{(1)}\big)^2 + g_1^{(1)}\Big)\Big(g_1^{(2)} + \frac{1}{2}\Big) .$$

For the 3-partition of size 2 $(\Box,\varnothing,\Box)$, we have $\mathsf{F}^G_{(\Box,\varnothing,\Box)} = \frac{9}{2}$ and $\mathsf{F}_{(\Box,\varnothing,\Box)} = 1$. We obtain, from Theorem 4 and using formula (46),

$$E_{\left(\boxed{1},\varnothing,\boxed{2}\right)} = \frac{2}{9}\, s_1(\mathrm{c}_2,0)\, s_1 \cdot \Big(\big(g_2^{(1)}\big)^2 - 1\Big)\Big(g_2^{(2)} - 1\Big) \cdot 3E_{\left(\boxed{1},\varnothing,\varnothing\right)}\Big|_{\mathrm{c}_2=0} .$$

In particular, Theorem 4 asserts that the above function is non-singular at $\mathrm{c}_2 = 0$ although, from (24), $s_1(\mathrm{c}_2,0)$ is singular at $\mathrm{c}_2 = 0$. As shows the proof of the Lemma 6, this comes from the fact that

$$e_{1,2} \cdot \Big(\big(g_2^{(1)}\big)^2 - 1\Big)\Big(g_2^{(2)} - 1\Big) \cdot E_{\left(\boxed{1},\varnothing,\varnothing\right)} = 0 .$$

So actually we have

$$\begin{aligned} E_{\left(\boxed{1},\varnothing,\boxed{2}\right)} &= \frac{2}{9}\Big(\big(g_2^{(1)}\big)^2 - 1\Big)\Big(g_2^{(2)} - 1\Big) \cdot 3E_{\left(\boxed{1},\varnothing,\varnothing\right)} \\ &= \frac{2}{9}\Big(\big(g_2^{(1)}\big)^2 - 1\Big)\Big(g_2^{(2)} - 1\Big)\Big(\big(g_1^{(1)}\big)^2 + g_1^{(1)}\Big)\Big(g_1^{(2)} + \frac{1}{2}\Big) . \end{aligned}$$

For the 3-partition of size 3 $(\Box\Box,\varnothing,\Box)$, we have $\mathsf{F}^G_{(\Box\Box,\varnothing,\Box)} = \frac{27}{2}$ and $\mathsf{F}_{(\Box\Box,\varnothing,\Box)} = 2$. We obtain, from Theorem 4 and using formula (46),

$$E_{\left(\boxed{1}\boxed{3},\varnothing,\boxed{2}\right)} = \frac{1}{27}\, s_2(1,0) s_1(1,0)\, s_1 s_2 \Big(\big(g_3^{(1)}\big)^2 + g_3^{(1)}\Big)\Big(g_3^{(2)} + \frac{1}{2}\Big) \cdot \frac{9}{2} E_{\left(\boxed{1},\varnothing,\boxed{2}\right)} .$$

Note that, as shows the proof of the Lemma 6, $e_{2,3} \cdot s_1(1,0)\, s_1 s_2 \Big(\big(g_3^{(1)}\big)^2 + g_3^{(1)}\Big)\Big(g_3^{(2)} + \frac{1}{2}\Big) \cdot E_{\left(\boxed{1},\varnothing,\boxed{2}\right)} = 0$, and so we have actually

$$E_{\left(\boxed{1}\boxed{3},\varnothing,\boxed{2}\right)} = \frac{1}{27}\, s_2 s_1(1,0)\, s_1 s_2 \Big(\big(g_3^{(1)}\big)^2 + g_3^{(1)}\Big)\Big(g_3^{(2)} + \frac{1}{2}\Big) \cdot \frac{9}{2} E_{\left(\boxed{1},\varnothing,\boxed{2}\right)} .$$

Remark. We notice that the central element $g^{(3)} = 1_G$ does not appear in the fusion formula (because $\xi_1^{(3)} = \xi_2^{(3)} = \xi_3^{(3)}$). Moreover, in the examples above, we could have used only the central element $g^{(1)}$ since its eigenvalues are enough to distinguish between the irreducible representations of G; namely, we would have obtained

$$E_{\left(\boxed{1},\varnothing,\varnothing\right)} = \frac{1}{2}\Big(\big(g_1^{(1)}\big)^2 + g_1^{(1)}\Big) ,$$

$$E_{\left(\boxed{1},\varnothing,\boxed{2}\right)} = -\frac{1}{2}\left(\left(g_2^{(1)}\right)^2-1\right)\left(\left(g_1^{(1)}\right)^2+g_1^{(1)}\right) ,$$

$$E_{\left(\boxed{1}\boxed{3},\varnothing,\boxed{2}\right)} = -\frac{1}{8}\, s_2s_1(1,0)\, s_1s_2\left(\left(g_3^{(1)}\right)^2+g_3^{(1)}\right)\left(\left(g_2^{(1)}\right)^2-1\right)\left(\left(g_1^{(1)}\right)^2+g_1^{(1)}\right) .$$

In general, the central element corresponding to the conjugacy class of the unit element of G never appears. Nevertheless, in general, all other central elements (2) of $\mathbb{C}G$ are necessary. In the following Section, we explain how to reduce the number of elements $g^{(\alpha)}$ appearing in the fusion procedure for the particular situation of an Abelian finite group G.

8. Simplified fusion formula when G is Abelian

From now let G be any finite Abelian group. It is a standard fact that G is thus isomorphic to a direct product of finite cyclic groups. So we assume that

$$G \cong C_{k_1}\times C_{k_2}\times\cdots\times C_{k_N} , \tag{56}$$

where N is a non-negative integer, $k_1,k_2,\dots,k_N\geq 2$ and C_{k_α} is the cyclic group of order k_α, $\alpha = 1,\dots,N$. Set $m := k_1k_2\dots k_N$, the cardinality of G.

As G is Abelian, the number of its conjugacy classes is equal to m. The set of central elements $g^{(\alpha)}$ of $\mathbb{C}G$ defined in (2) coincide here with the set of elements of G. All these elements appear in the fusion formula of the previous Section, see (41). We will provide a simplified formula using a certain subset of elements of G.

For $\alpha = 1,\dots,N$, we choose and denote by $t^{(\alpha)}$ an element of G which generates the cyclic group C_{k_α} appearing in (56). Notice that the elements $t^{(\alpha)}$, $\alpha = 1,\dots,N$, satisfy the relations

$$\left(t^{(\alpha)}\right)^{k_\alpha}=1 \quad\text{ and }\quad t^{(\alpha)}t^{(\alpha')}=t^{(\alpha')}t^{(\alpha)} \qquad \text{for } \alpha,\alpha'=1,\dots,N.$$

Moreover the elements $t^{(\alpha)}$, $\alpha = 1,\dots,N$, generate the group G and thus the knowledge of their eigenvalues is sufficient to distinguish between the irreducible representations of G. Namely, any irreducible representation of G is one-dimensional and is obtained by sending $t^{(\alpha)}$ to a k_α-th root of unity for $\alpha = 1,\dots,N$ (there are m non-isomorphic representations of this sort and they exhaust the set of irreducible representations of G).

So now let $S^{(\alpha)} := \{\xi_1^{(\alpha)},\dots,\xi_{k_\alpha}^{(\alpha)}\}$ be the set of all k_α-th roots of unity, and, for $i = 1,\dots,n$, define as in (12)

$$t_i^{(\alpha)}(v) := \frac{\prod_{\xi^{(\alpha)}\in S^{(\alpha)}}(v-\xi^{(\alpha)})}{v-t_i^{(\alpha)}} \qquad \text{for } \alpha=1,\dots,N , \tag{57}$$

where we recall that $t_i^{(\alpha)}$, $i = 1,\dots,n$, is the image of the element $t^{(\alpha)}$ by the injective morphism $\iota\circ\varpi_i$ from $\mathbb{C}G$ to $\mathbb{C}\widetilde{G}_n$, see (11).

From (6), it is straightforward to see that we have, for $i = 1,\dots,n$ and $\alpha = 1,\dots,N$,

$$t_i^{(\alpha)}(v) = v^{k_\alpha-1}+v^{k_\alpha-2}t_i^{(\alpha)}+\cdots+v\left(t_i^{(\alpha)}\right)^{k_\alpha-2}+\left(t_i^{(\alpha)}\right)^{k_\alpha-1} .$$

We define

$$\Gamma'(\underline{v_1},\dots,\underline{v_n}) := \prod_{i=1}^{n}\prod_{\alpha=1}^{N} t_i^{(\alpha)}(v_i^{(\alpha)}) \ , \tag{58}$$

and, for an m-partition $\underline{\lambda}$ (compare with (28)),

$$\mathsf{F'}^G_{\underline{\lambda}} := \prod_{\underline{\theta}\in\underline{\lambda}}\Bigg(\prod_{\alpha=1}^{N}\Big(\prod_{\substack{\xi^{(\alpha)}\,\in\, S^{(\alpha)}\\ \xi^{(\alpha)}\,\neq\,\xi^{(\alpha)}_{\mathrm{p}(\underline{\theta})}}} (\xi^{(\alpha)}_{\mathrm{p}(\underline{\theta})} - \xi^{(\alpha)})\ \Big)\Bigg)\ , \tag{59}$$

The formulation of the Theorem 4 remains the same with $\Gamma(\underline{v_1},\dots,\underline{v_n})$ replaced by $\Gamma'(\underline{v_1},\dots,\underline{v_n})$ in the defining formula (43) for the function $\Phi(u_1,\dots,u_n,\underline{v_1},\dots,\underline{v_n})$, and $\mathsf{F}^G_{\underline{\lambda}}$ replaced by $\mathsf{F'}^G_{\underline{\lambda}}$.

Remark. Let ζ be a k-th root of unity. Recall that $\prod_{\xi}(\zeta-\xi)=k\zeta^{-1}$, where the product is taken on the k-th roots of unity ξ different from ζ. Thus, for G as in (56), we have for the coefficients $\mathsf{F'}^G_{\underline{\lambda}}$ defined by (59):

$$\mathsf{F'}^G_{\underline{\lambda}} = \prod_{\underline{\theta}\in\underline{\lambda}}\Big(\prod_{\alpha=1}^{N} \frac{k_\alpha}{\xi^{(\alpha)}_{\mathrm{p}(\underline{\theta})}}\Big)\ .$$

Example. Consider the situation $N=1$ and $k_1=3$ in (56). Thus G is the cyclic group of order 3, and $\widetilde{G}_n$ is the complex reflection group of type $G(3,1,n)$. Let $t:=t^{(1)}$ and let $\{\xi_1,\xi_2,\xi_3\}$ be the set of all third roots of unity. The Theorem 4 (in its simplified version explained above) asserts that, for example, the idempotent $E_{\left(\boxed{1}\boxed{3},\varnothing,\boxed{2}\right)}$ can be expressed as

$$E_{\left(\boxed{1}\boxed{3},\varnothing,\boxed{2}\right)} = \frac{\xi_1^2\xi_3}{54} s_2(1,0)s_1(1,0)s_1s_2s_1(0,0)s_1\,(\xi_1^2+\xi_1t_3+t_3^2)(\xi_3^2+\xi_3t_2+t_2^2)(\xi_1^2+\xi_1t_1+t_1^2)\,.$$

As already seen in the examples of the preceding Section, the proof of the Theorem 4 shows that we actually have:

$$E_{\left(\boxed{1}\boxed{3},\varnothing,\boxed{2}\right)} = \frac{\xi_1^2\xi_3}{54} s_2s_1(1,0)s_1s_2\,(\xi_1^2+\xi_1t_3+t_3^2)(\xi_3^2+\xi_3t_2+t_2^2)(\xi_1^2+\xi_1t_1+t_1^2)\,.$$

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